

Chaotic quantum decay in driven biased optical lattices

S. MOSSMANN^{1,2}, C. SCHUMANN¹ and H. J. KORSCH¹(*)

¹ *Fachbereich Physik, Technische Universität Kaiserslautern, D-67653 Kaiserslautern, Germany*

² *present address: Centro de Ciencias Físicas, UNAM, México*

PACS. 03.65.-w – Quantum mechanics.

PACS. 03.65.Yz – Decoherence; open systems; quantum statistical methods.

PACS. 05.45.-a – Chaotic and nonlinear dynamical systems.

Abstract. – Quantum decay in an ac driven biased periodic potential modeling cold atoms in optical lattices is studied for a symmetry broken driving. For the case of fully chaotic classical dynamics the classical exponential decay is quantum mechanically suppressed for a driving frequency ω in resonance with the Bloch frequency ω_B , $q\omega = r\omega_B$ with integers q and r . Asymptotically an algebraic decay $\sim t^{-\gamma}$ is observed. For $r = 1$ the exponent γ agrees with q as predicted by non-Hermitian random matrix theory for q decay channels. The time dependence of the survival probability can be well described by random matrix theory. The frequency dependence of the survival probability shows pronounced resonance peaks with sub-Fourier character.

Introduction. – Chaotic classical dynamics has quantum signatures in statistical properties of eigenvalues, wave amplitudes and state projections [1] which can be described by random matrix theory. This has been demonstrated in many cases for bound systems. For open systems, however, such studies are rare. Only recently, the statistics of the lifetimes of the metastable resonance states (or the complex part Γ of the resonance energies) have been shown to be in agreement with the appropriate random matrix results. This has been achieved for only few physical systems, e.g. open billiards [2–4], scattering on graphs [5] and strongly ac driven periodic lattices [6, 7] which will also be studied in the present letter.

Open classical systems with fully chaotic intrinsic dynamics show generically an exponential decay of the survival probability $P(t) = e^{-\nu t}$ in the long time limit, where the decay rate ν can be related to the Lyapunov exponent and the fractal dimension of the chaotic repeller [10]. Quantum mechanically, the situation is more complicated because the number of open decay channels depends on the Planck constant \hbar and typically increases in the limit of small \hbar .

(*) E-mail: korsch@physik.uni-kl.de

The model. – The one-dimensional Hamiltonian

$$H = p^2/2 + V(x) + F(t)x, \quad V(x + 2\pi) = V(x), \quad (1)$$

with space period $d = 2\pi$ and ac-dc driving

$$F(t) = F_0 + F_\omega(t), \quad F_\omega(t + T_\omega) = F_\omega(t) \quad (2)$$

(where the time-average of $F_\omega(t)$ can be chosen to be zero) is known as the ac-dc Wannier-Stark system. Such systems have been studied by many authors in different context (see, e.g. the review [7]). In recent years, one could observe an increasing interest in view of the exploding studies on the dynamics of cold atomic gases or Bose-Einstein condensates in optical lattices, where the static field F_0 is generated by chirping the laser frequencies or simply by the gravitational force.

Note that in the scaled units used here, \hbar depends on the system parameters and can be expressed as $\hbar = 4\sqrt{E_R/V_0}$ where V_0 is the potential depth and $E_R = \hbar^2 k^2 / (2m)$ is the recoil energy [7]. In present experiments a ratio of $V_0/E_R \approx 500$ is routinely available which yields $\hbar \approx 0.2$ not far from the value $\hbar = 0.1$ used in the numerical calculations below.

It is convenient to rewrite the dynamics in a Kramers-Henneberger form [7]

$$H_{\text{KH}} = (p - F_0 t)^2/2 + V(x - K_\omega(t)) \quad (3)$$

with $K_\omega(t) = \int_{t_0}^t dt' G_\omega(t')$, $G_\omega(t) = \int_{t_0}^t dt' F_\omega(t')$. We will study the potential $V(x) = \cos x$ with de-symmetrized driving

$$F_\omega(t) = A_\omega (\cos \omega t + \sin 2\omega t) \quad (4)$$

and we choose the time t_0 as a solution of $2 \sin \omega t_0 = \cos 2\omega t_0$ with the consequence that

$$K_\omega(t) = -\frac{\epsilon}{4} (4 \cos \omega t + \sin 2\omega t - 4 \cos \omega t_0 - \sin 2\omega t_0) \quad (5)$$

($\epsilon = A_\omega/\omega^2$) is periodic.

Chaotic dynamics. – Let us first consider the classical dynamics. A stroboscopic phase space plot for the non-biased case $F_0 = 0$, e.g. for parameters $\omega = 1$, $\epsilon = 3$, reveals a chaotic strip with few embedded regular islands. When a weak static force F_0 is added, this chaotic strip survives but the invariant curves confining the chaotic strip are destroyed. A trajectory starting inside the strip will show a diffusive motion until it reaches the (former) boundary p_1 where it escapes to infinity. As a measure of the local spreading in phase space we calculate the stability (or monodromy) matrix $M(t)$ ⁽¹⁾ for a trajectory started at a phase space point (x, p) . Figure 1 shows the norm $\|M(t)\|$ at time $t = 6T_\omega$ as a function of (x, p) for $\omega = 1$, $\epsilon = 3$ and an additional static force $F_0 = 0.016$, a value used also in the quantum case discussed below for resonant driving. We clearly observe the chaotic strip where the lower boundary at $p_1 \approx -4$ is relatively sharp whereas the upper boundary at $p_2 \approx +6$ is more diffusive. This difference is due to the fact that particles with large enough negative momentum escape and particles with positive momentum are reflected by the increasing linear potential and re-injected into the chaotic strip. A similar behavior is found for other values of F_0 used in the quantum calculations discussed in the following.

⁽¹⁾For a bound system, the long time average of the norm of the stability matrix gives the Lyapunov exponent. Here the majority of the trajectories escape to infinity and this limit is not defined. For more information on phase space delocalization see [8].

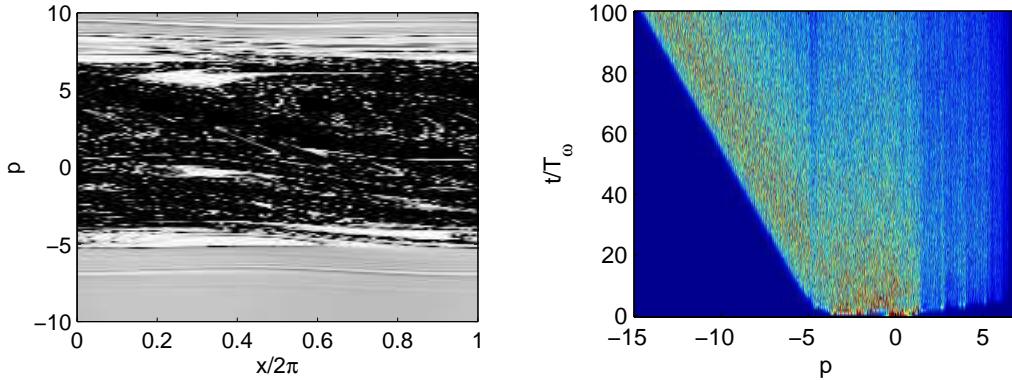


Fig. 1 – Norm of the monodromy matrix as a function of the initial phase space point (x, p) for a time propagation over six periods (left) and time evolution of a wavefunction $|\psi(p, t)|$ initially localized in the chaotic strip (right). The parameters are $\omega = 1$, $\epsilon = 3$ with a static field is $F_0 = 0.016$.

The quantum dynamics is qualitatively different because of the Bloch oscillation [9], an intrinsic periodic motion with the Bloch period in the non-driven case $F_\omega(t) = 0$. By constructing a generalized scattering-matrix for such systems, resonance states can be defined and also easily computed [7], provided that the driving and the Bloch frequency $\omega_B = dF_0/\hbar$ are in resonance,

$$q\omega = r\omega_B, \quad (6)$$

where r and q are coprime integers. The integer q is equal to the dimension of the scattering matrix, i.e. the number of open channels [7]. This offers the unique possibility to tune the number of decay channels by varying the external system parameters.

Here we will explore the quantum decay dynamics for $r = 1$ as a function of the number q of decay channels, starting from the case of resonant driving, $\omega = \omega_B$, i.e. $q = 1$. Here and in the following we use $\hbar = 0.1$. The quantum wavefunction $|\psi(p, t)|$ is plotted in fig. 1 for an initial minimum uncertainty Gaussian wavepacket with momentum width 0.28 centered at $p = 0$ in the classically chaotic region. Also here the chaotic strip can be easily identified: The wavefunction rapidly spreads over the strip and decays in the direction of negative momentum. Inside the strip we find an irregular oscillation and outside, for $p < p_1$, a decay to $-\infty$ with linearly growing momentum $p \sim -F_0 t$. If the assumption of an irregular wavefunction inside the chaotic strip is correct, the statistics of the normalized probabilities $s_n = |\psi(p_n, t)|^2 / \sum_n |\psi(p_n, t)|^2$ at discrete values of the momentum should follow the random vector model [1, 12, 13] describing in general the fluctuations of a sum of ν independent Gaussian distributed variables ($\nu = 1, 2, 4$) for the orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensemble, respectively. This model predicts approximately a χ_ν^2 distribution

$$W_\nu(s) \approx \chi_\nu^2(s) = \left(\frac{\nu}{2}\right)^{\nu/2} \frac{s^{\nu/2-1}}{\Gamma(\nu/2)} e^{-\nu s/2}. \quad (7)$$

Figure 2 shows a comparison of the numerical data of fig. 1 in the chaotic strip for $t = 100 T_\omega$ with the distributions (7) for $\nu = 1, 2, 4$. As expected, we observe an agreement with the GUE statistics. Moreover, this behavior is found to be independent of the ratio $\omega/\omega_B = 1/q$. The decay, however, is very sensitive with respect to this ratio.

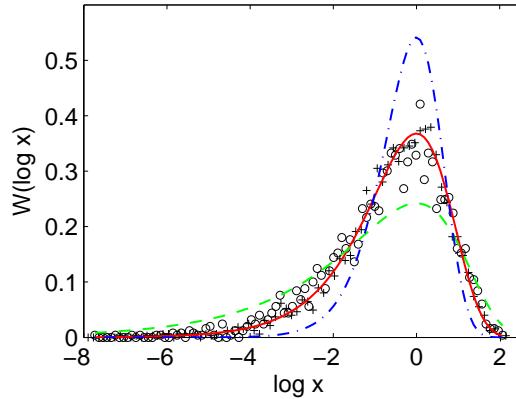


Fig. 2 – Statistical distribution of $s = |\psi(p, 100 T_\omega)|^2$ within the chaotic strip for $q=1$ (+) and $q=2$ (o) in comparison with the random matrix predictions (7) for the GOE (- · - ·), GUE (—) and the GSE (- - -) case. Shown is the distribution $W(\log x)$ with $x = s/\bar{s}$ in order to increase the differences.

Dynamics of decay. – In the following, we will explore the validity of the conjecture that the decay of these Wannier-Stark systems can be described by non-Hermitian random matrix theory for systems with q decay channels.

According to random matrix theory, the distributions of the resonance widths Γ of the Wannier-Stark resonance states (the eigenstates of the Floquet operator) are given by [14]

$$\Pi(\Gamma_s) = \frac{(-1)^q}{(q-1)!} \Gamma_s^{q-1} \frac{d^q}{d\Gamma_s^q} \left[\frac{1 - \exp(-2\Gamma_s)}{2\Gamma_s} \right], \quad (8)$$

for the circular unitary ensemble (CUE). Here q is the number of channels and $\Gamma_s = \pi\Gamma/\Delta$ is the width scaled by the mean level spacing Δ . Equation (8) can also be applied to the case of a harmonic driving as shown in [7] if one restricts the Hamiltonian to a fixed value of the quasimomentum κ which is a good quantum number. With the exception of the center and the boundaries of the Brillouin zone the time reversal symmetry is then broken.

A random matrix formula for the time dependence of the decay of an open quantum system has been derived by Savin and Sokolov [15] using supersymmetry techniques. For CUE the case of perfect coupling is realized [7] (i.e. one has $T = 1$ in eq. (11) of ref. [15]) and the decay probability depends only on the number of channels q and the ratio $\tau = t/T_H$ where T_H is the Heisenberg time $T_H = 2\pi\hbar/\Delta$:

$$P^{(q)}(t) = \frac{1}{2} \int_{-1}^1 du \int_1^\infty dv \frac{v+u}{v-u} \delta(2\tau+u-v) \left[\frac{1+u}{1+v} \right]^q. \quad (9)$$

The integral can be expressed in terms of hypergeometric functions as

$$P^{(q)}(t) = (1+\tau)^{-q} f_{\leq}(t) \quad \text{for } \tau \leq 1, \quad (10)$$

$$\begin{aligned} f_{\leq}(\tau) = & \sum_{k=0}^q \binom{q}{k} (-\tau)^k \left[\frac{1+\tau}{k+1} {}_2F_1 \left(q, k+1; k+2; \frac{\tau}{1+\tau} \right) \right. \\ & \left. - \frac{2\tau}{k+2} {}_2F_1 \left(q, k+2; k+3; \frac{\tau}{1+\tau} \right) \right], \end{aligned} \quad (11)$$

$$f_{>}(\tau) = \frac{1+1/\tau}{q+1} {}_2F_1\left(q, 1; q+2; \frac{1}{1+\tau}\right) - \frac{2/\tau}{(q+1)(q+2)} {}_2F_1\left(q, 2; q+3; \frac{1}{1+\tau}\right). \quad (12)$$

Asymptotically the decay is algebraic,

$$P^{(q)}(t) \rightarrow \tau^{-q}/(q+1) \quad \text{for } \tau \rightarrow \infty, \quad (13)$$

where the exponent is equal to the number of decay channels.

Quantum stabilization, i.e. algebraic decay was already observed in previous studies for a purely harmonic driving (see [16] and references therein) but with markedly different decay exponents. For the symmetry broken driving (4) (and therefore with proper CUE symmetry contrary to the harmonic case) also good quantitative agreement with the channel number q is found as we will show in the following.

The system parameters are the same as in fig. 1, the static force F_0 is tuned to give a rational ratio $\omega/\omega_B = r/q$ with $r = 1$. A minimum uncertainty wavepacket with momentum width 0.28 centered at $p = 0$ and $x = \pi$ is propagated in time and the survival probability in the chaotic strip $p_1 < p < p_2$ is calculated from the momentum distribution,

$$P(t) = \int_{p_1}^{p_2} dp |\psi(p, t)|^2, \quad (14)$$

where $p_1 = -5$ and $p_2 = 7$ are chosen based on fig. 1. Numerically, these survival probabilities are calculated up to a time $t = 200 T_\omega$ for $q = 1, 2, 3$ and 4.

A fit of the long time behavior of the numerical data to the asymptotics (13) confirms the algebraic decay predicted by random matrix theory: Numerically the exponents are calculated as 0.999, 2.006, 3.094 and 4.087 for $q = 1, 2, 3$ and 4 in precise agreement with these channel numbers (compare the double logarithmic plot in fig. 3). The corresponding Heisenberg times are found to be $T_H/T_\omega = 53.2, 36.5, 30.0$ and 25.3 and decrease approximately as $\sim q^{-1/2}$. For resonant driving (6) with $r \neq 1$ the decay is also found to be asymptotically algebraic, however

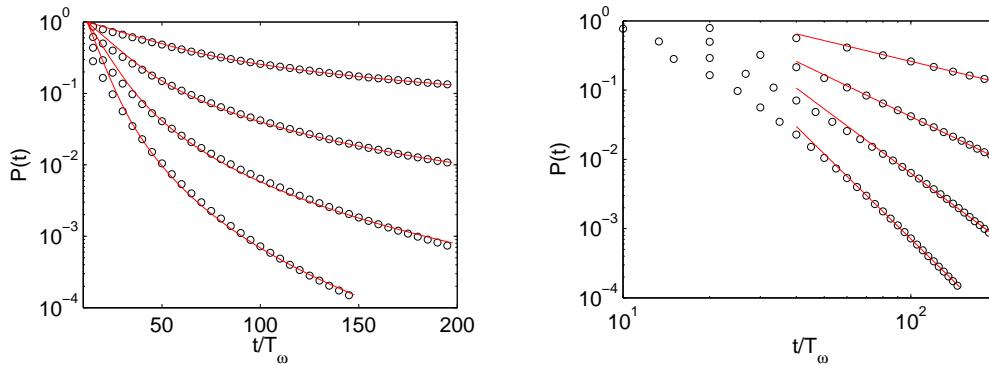


Fig. 3 – Decay of the quantum survival probability $P(t)$ for rational frequency ratios $\omega/\omega_B = 1/q$ with $q = 1, 2, 3, 4$ (open circles, from top to bottom) in comparison with the results given by non-Hermitian random matrix theory for q decay channels (solid lines). Right: double logarithmic plots with asymptotic linear fits.

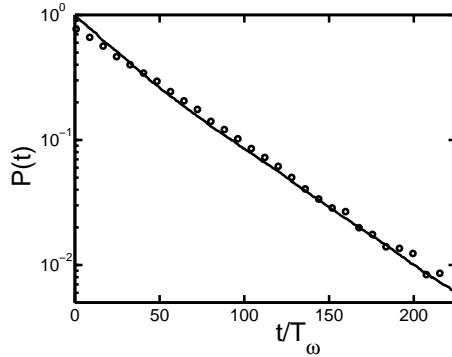


Fig. 4 – Quantum survival probability $P(t)$ for irrational frequency ratios $\omega/\omega_B = 1/\sqrt{2}$ (full curve) in comparison with the results from a corresponding classical ensemble (open circles).

with exponents differing from q which is presumably due to the fact that the quasienergy spectrum is r -fold degenerate which introduces additional symmetries into the system.

Figure 3 shows a logarithmic plot of the numerical results for the survival probabilities (14) as a function of t/T_ω up to $t = 200T_\omega$ for rational ratios $\omega/\omega_B = 1/q$ for $q = 1, 2, 3$ and 4. Also shown in fig. 3 is the time dependence of the random matrix model (10) using the values of the Heisenberg time extracted by the asymptotic fit. In order to account for the initial spreading of the wavepacket to equilibrate over the chaotic strip, the numerical data are shifted by $8T_\omega$ in each case. Good agreement is observed. The small deviations between the numerical data and the random matrix predictions (in particular for $q = 3$) are due to the influence of small stability islands in the chaotic strip.

For an irrational ratio ω/ω_B the quantum survival probability decays exponentially, $P(t) = e^{-\nu t}$, as the classical distribution and even the exponent ν agrees with the classical one. A comparison is given in fig. 4. As shown by Puhlmann et al. [17], the random matrix predictions for the survival probability can also be analyzed using a semiclassical approach. A semiclassical treatment of the driven Wannier-Stark system along these lines, however, deserves further studies.

Resonance profiles. – It is obvious that the quantum survival probabilities $P(t)$ depend very sensitively on the frequency ratio ω/ω_B . Asymptotically, the decay is algebraic for rational and exponential for irrational ratios and therefore a fractal-like spike structure develops as already discussed in ref. [11]. As a typical example, fig. 5 shows $P(t)$ as a function of ω/ω_B in the vicinity of the resonance $\omega = \omega_B$ for $t = 300T_\omega$. We observe a pronounced resonance peak which sharpens with increasing time. The maximum decays as $\sim t^{-1}$ and the almost constant background decays exponentially. The peak profiles can be well reproduced by the functional form $P_{\text{fit}}(t) = a/\sqrt{(\omega - \omega_B)^2 + 3(\Delta\omega)^2} + b$ also shown in the figure. The parameters a, b and $\Delta\omega$ are functions of time. The time decay of the width $\Delta\omega$ of the distribution was found to be algebraic, $\Delta\omega \sim t^{-\gamma}$, with $\gamma \approx 1.45$. Interestingly, this is much faster than the Fourier behavior where two frequencies can only be distinguished if their distance is at least of the order of the excitation time, i.e. $\Delta\omega \sim t^{-1}$. Similar observations of such ‘sub-Fourier’ resonances have been reported recently in an experimental study of bichromatically kicked cold atoms in an optical lattice [18,19]. This system also shows quantum chaotic behavior and the mechanism of the sub-Fourier sharpening of these resonances could be explained theoretically [20] based on the phenomenon of dynamical localization.

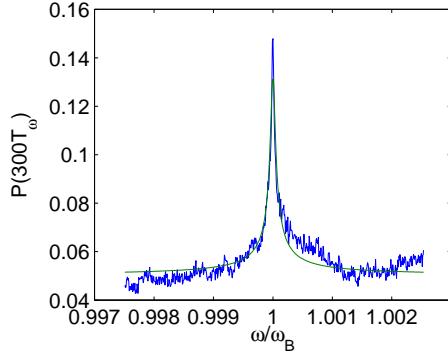


Fig. 5 – Survival probability $P(t)$ as a function of ω/ω_B in the vicinity of the resonance $\omega = \omega_B$ for $t = 300T_\omega$.

Support from the Deutsche Forschungsgemeinschaft (GRK 792) and from the Volkswagen-Stiftung is gratefully acknowledged. S.M. acknowledges the support by a fellowship within the Postdoc-Programme of the the German Academic Exchange Service (DAAD). We would also like to thank Jean-Claude Garreau and Dmitry Savin for discussions and Dominique Delande for providing a copy of ref. [20] prior to publication.

REFERENCES

- [1] HAAKE F., *Quantum Signatures of Chaos* (Springer, Berlin, Heidelberg, New York) 2001
- [2] ISHIO H. and BURGDÖRFER J., *Phys. Rev. B*, **51** (1995) R2013
- [3] WIRTZ L., TANG J.-Z. and BURGDÖRFER J., *Phys. Rev. B*, **56** (1997) 7589
- [4] ISHIO H., *Phys. Rev. E*, **62** (2000) R3035
- [5] KOTTOS T. and SMILANSKY U., *Phys. Rev. Lett.*, **79** (1997) 4794; *Phys. Rev. Lett.*, **85** (2000) 968
- [6] GLÜCK M., KOLOVSKY A. R. and KORSCH H. J., *Phys. Rev. Lett.*, **82** (1999) 1534; *Phys. Rev. E*, **60** (1999) 247
- [7] GLÜCK M., KOLOVSKY A. R. and KORSCH H.J., *Phys. Rep.*, **366** (2002) 103
- [8] KORSCH H. J. AND LEYES W., *New J. Phys.*, **4** (2002) 62
- [9] HARTMANN T., KECK F., KORSCH H. J. and S. MOSSMANN S., *New J. Phys.*, **6** (2004) 2
- [10] GASPARD P., *Chaos, Scattering, and Statistical Mechanics* (Cambridge University Press, Cambridge) 1998
- [11] GLÜCK M., KOLOVSKY A.R. and KORSCH H. J., *Europhys. Lett.*, **51** (2000) 255
- [12] MIRBACH B. and KORSCH H. J., *Ann. Phys. (N.Y.)*, **265** (1998) 80
- [13] ZIMMERMANN T., KÖPPEL H. and CEDERBAUM L. S., *J. Chem. Phys.*, **91** (1989) 3934
- [14] FYODOROV Y. V. and SOMMERS H.-J., *JETP Lett.*, **63** (1996) 1026; *J. Math. Phys.*, **38** (1997) 1918
- [15] SAVIN D. V. and SOKOLOV V. V., *Phys. Rev. E*, **56** (1997) R4911
- [16] GLÜCK M., KOLOVSKY A.R. and KORSCH H. J., *Physica E*, **9** (2001) 478
- [17] PUHLMANN, H., SCHANZ, H., KOTTOS, T. and GEISEL, T., *Europhys. Lett.*, **69** (2005) 313
- [18] SZRIFTGISER P., RINGOT J., DELANDE D. and GARREAU J. C., *Phys. Rev. Lett.*, **89** (2002) 224101

- [19] SZRIFTGILER P., LIGNIER H., RINGOT J., GARREAU J. C. and DELANDE D., *Commun. Nonlin. Sci. Num. Simul.*, **8** (2003) 301
- [20] LIGNIER H., GARREAU J.-C., SZRIFTGILER P. and DELANDE D., *Europhys. Lett.*, **69** (2005) 327